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Transitions between ordered phases

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Abstract. A Hamiltonian which shows two successive phase transitions on cooling is studied by the $1/n$ method correct to order $1/n^2$. Despite the presence of Goldstone modes in the phase which orders first, the second phase transition is found to have 'ordinary' critical exponents determined solely by the components of the order parameter that go soft at the second transition.

1. Introduction

Phase transitions between a disordered phase and an ordered phase have been the subject of extensive study by renormalization group methods. (See, for example, the review article by Fisher (1974).) However, phase transitions between two ordered phases have not so far been given much attention.

If on cooling from the disordered phase two successive phase transitions occur, and if the first of these is Ising-like, then we may expect the nature of the second transition to be determined by the components of the order parameter which are going soft at that transition. If, on the other hand, the first transition is Heisenberg-like, in the sense that the ordering produces Goldstone bosons, then it is not clear whether or not the nature of the second transition will be modified by the long range forces due to the massless Goldstone modes. This is the question we shall investigate by using the $1/n$ expansion. The reason for using the $1/n$ expansion instead of the ϵ expansion is that the one-loop diagrams for the inverse longitudinal susceptibility have an infrared divergence which is removed by summing the complete chain of bubbles which occurs in the $1/n$ expansion.

The model we use for the discussion has the following free energy functional:

$$F=\tfrac{1}{2}r_1\boldsymbol{x}^2+\tfrac{1}{2}r_2\boldsymbol{y}^2+\tfrac{1}{2}(\partial\boldsymbol{x})^2+\tfrac{1}{2}(\partial\boldsymbol{y})^2+\tfrac{1}{24}u(\boldsymbol{x}^2)^2+\tfrac{1}{24}v(\boldsymbol{y}^2)^2+\tfrac{1}{4}w\boldsymbol{x}^2\boldsymbol{y}^2 \qquad (1.1)$$

where $\boldsymbol{x}$ has m components, and $\boldsymbol{y}$ has n components, and r_1 and r_2 are linear in temperature. This model has been considered by Kosterlitz *et al* (1976), and by Bruce and Aharony (1975), in connection with bicritical and tetracritical points, using the renormalization group to calculate the nature of the transition from the disordered phase to the ordered phases. It has also been discussed in mean-field theory for all values of the parameters r_1, r_2, u, v and w by Imry (1975).

The model Hamiltonian describes a number of real systems, for example, uniaxial antiferromagnets in appropriately aligned fields (Kosterlitz *et al* 1976). However, in that case the multicritical point is bicritical and there are only two ordered phases (x-ordered and y-ordered) corresponding to the antiferromagnetic and spin-flopped phases separated by a first-order line. For displacive phase transitions (Bruce and Aharony 1975) the multicritical point may be tetracritical, and in that case there is also a phase with mixed $x-y$ ordering. It is this latter situation which is being considered here. We shall not be interested in the vicinity of the tetracritical point but shall discuss the model for choices of the parameters such that the $T-w$ phase diagram is of the form shown in figure 1, with the two phase transitions well separated in temperature.

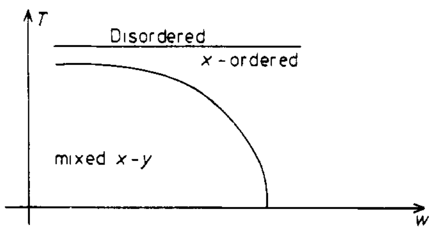

Figure 1. Phase diagram.

In the x-ordered phase the order parameter is of the form

$$\boldsymbol{x}=(0,0,\ldots,\rho), \qquad \boldsymbol{y}=(0,0,\ldots,0) \tag{1.2}$$

and in the mixed $x-y$ phase it is of the form

$$\boldsymbol{x}=(0,0,\ldots,\rho), \qquad \boldsymbol{y}=(0,0,\ldots,\sigma). \tag{1.3}$$

In mean-field theory the mixed $x-y$ phase forms when $r_2-3wr_1/u\leqslant 0$ and in the mixed $x-y$ phase

$$\rho^2=2v(r_1-3wr_2/v)/3(w^2-\tfrac{1}{9}uv)$$

and

$$\sigma^2=2u(r_2-3wr_1/u)/3(w^2-\tfrac{1}{9}uv)$$

where $w^2<\frac{1}{9}uv$.

2. Longitudinal propagators

In the mixed $x-y$ phase, it is convenient to perform the shift

$$\begin{aligned}\boldsymbol{x}&\to(\boldsymbol{x}_\perp, x_m)+(0,0,\ldots,\rho)\\ \boldsymbol{y}&\to(\boldsymbol{y}_\perp, y_n)+(0,0,\ldots,\sigma)\end{aligned} \tag{2.1}$$

where $\boldsymbol{x}_\perp$ is the first $m-1$ components of the shifted $\boldsymbol{x}$ field, $\boldsymbol{y}_\perp$ is the first $n-1$ components of the shifted $\boldsymbol{y}$ field, and ρ and σ are the exact order parameters. The free energy functional of equation (1.1) then becomes

$$F=\tfrac{1}{2}[(\partial \boldsymbol{x}_\perp)^2+(\partial \boldsymbol{y}_\perp)^2+(\partial x_m)^2+(\partial y_n)^2]+\tfrac{1}{2}r_T\boldsymbol{x}_\perp^2+\tfrac{1}{2}\tilde{r}_T\boldsymbol{y}_\perp^2$$
$$+\tfrac{1}{2}r_Lx_m^2+\tfrac{1}{2}\tilde{r}_Ly_n^2+r_{xy}x_my_n+\tfrac{1}{2}\boldsymbol{x}_\perp^2(r_1+\tfrac{1}{6}\rho^2u+\tfrac{1}{2}\sigma^2w-r_T)$$
$$+\tfrac{1}{2}\boldsymbol{y}_\perp^2(r_2+\tfrac{1}{6}\sigma^2v+\tfrac{1}{2}\rho^2w-\tilde{r}_T)+\tfrac{1}{2}x_m^2(r_1+\tfrac{1}{2}\rho^2u+\tfrac{1}{2}\sigma^2w-r_L)$$
$$+\tfrac{1}{2}y_n^2(r_2+\tfrac{1}{2}\sigma^2v+\tfrac{1}{2}\rho^2w-\tilde{r}_L)+\rho\sigma w x_my_n+\tfrac{1}{24}u(\boldsymbol{x}^2)^2+\tfrac{1}{24}v(\boldsymbol{y}^2)^2$$
$$+\tfrac{1}{4}w\boldsymbol{x}^2\boldsymbol{y}^2+\tfrac{1}{6}\rho u\boldsymbol{x}^2x_m+\tfrac{1}{6}\sigma v\boldsymbol{y}^2y_n+\tfrac{1}{2}\sigma w\boldsymbol{x}^2y_n+\tfrac{1}{2}\rho w\boldsymbol{y}^2x_m$$
$$+(\rho r_1+\tfrac{1}{6}\rho^3u+\tfrac{1}{2}\rho\sigma^2w-h_\rho)x_m+(\sigma r_2+\tfrac{1}{6}\sigma^3v+\tfrac{1}{2}\sigma\rho^2w-h_\sigma)y_n \tag{2.2}$$

where h_ρ and h_σ are the ordering fields for $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively, and we have introduced 'renormalized masses' r_T, $\tilde{r}_T$, r_L, $\tilde{r}_L$, and r_{xy}. It should be observed that x_m and y_n are not mass eigenstates.

The limit of large m and n will be taken in such a way that m/n remains finite, and the couplings u, v and w will be taken of order $1/n$, and the order parameters ρ and σ of order $\sqrt{n}$. In § 3, the equation of state for the mixed x–y phase will be calculated to next to leading order in $1/n$, and this will require a knowledge of the longitudinal propagators in zeroth order in $1/n$.

Let us denote by $C(p)$ the sum of chains of $\boldsymbol{x}_\perp$ bubbles with at least one bubble (see figure 2(a)):

$$C(p)=mI(p,r_T)/2(1+\tfrac{1}{6}muI(p,r_T)) \tag{2.3}$$

where

$$I(p,r)=\int \mathrm{d}^dk/(2\pi)^d[(p+k)^2+r](k^2+r). \tag{2.4}$$

Similarly, let us denote by $D(p)$ the sum of chains of $\boldsymbol{y}_\perp$ bubbles with at least one bubble (see figure 2(b)):

$$D(p)=nI(p,\tilde{r}_T)/2(1+\tfrac{1}{6}nvI(p,\tilde{r}_T)). \tag{2.5}$$

Next, let us denote by T_{xx}, T_{xy}, T_{yx} and T_{yy} the sums of chains of bubbles of either type ($\boldsymbol{x}_\perp$ or $\boldsymbol{y}_\perp$) beginning in an x bubble and ending in an x bubble, beginning in an x bubble and ending in a y bubble, beginning in a y bubble and ending in an x bubble, and beginning in a y bubble and ending in a y bubble, respectively.

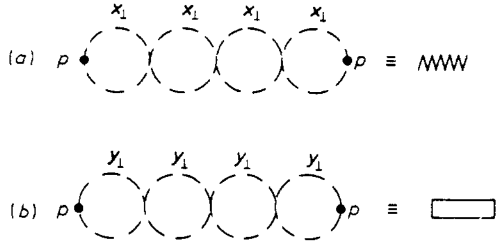

Figure 2. Pure chains of x bubbles, and pure chains of y bubbles.

From figure 3, it can be seen that

$$\begin{aligned} T_{xx} &= C/(1-w^2CD) \\ T_{xy} &= T_{yx} = -wCD/(1-w^2CD) \\ T_{yy} &= D/(1-w^2CD). \end{aligned} \tag{2.6}$$

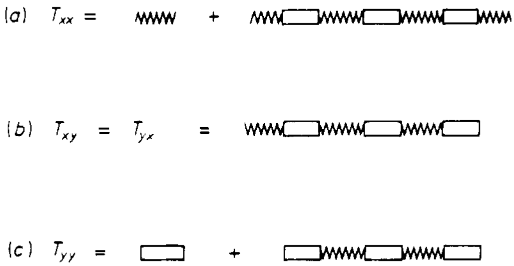

Figure 3. Chains of bubbles ending in bubbles of definite type.

The self-energy parts for x_m and y_n, Σ_x and Σ_y, and the off-diagonal self-energy part connecting x_m to y_n, Σ_{xy}, may be constructed from T_{xx}, T_{xy}, T_{yx} and T_{yy}, and the one-loop diagrams of figure 4. Thus,

$$\Sigma_x = (\tfrac{1}{3}\rho u)^2 T_{xx} + (\rho w)^2 T_{yy} + (\tfrac{2}{3}\rho^2 uw)T_{xy} - \tfrac{1}{6}umK(r_T) - \tfrac{1}{2}wnK(\tilde{r}_T) \tag{2.7}$$

where

$$K(r) = \int d^dk/(2\pi)^d(k^2+r). \tag{2.8}$$

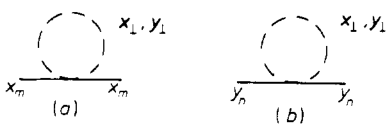

Figure 4. One-loop diagrams for longitudinal masses.

Substituting from equation (2.6) in (2.7) gives

$$\Sigma_x = \rho^2(\tfrac{1}{9}u^2C + w^2D - \tfrac{2}{3}w^2uCD)/(1-w^2CD) - \tfrac{1}{6}umK(r_T) - \tfrac{1}{2}wnK(\tilde{r}_T). \tag{2.9}$$

Similarly,

$$\Sigma_y = \sigma^2(\tfrac{1}{9}v^2D + w^2C - \tfrac{2}{3}w^2vCD)/(1-w^2CD) - \tfrac{1}{6}vnK(\tilde{r}_T) - \tfrac{1}{2}wmK(r_T) \tag{2.10}$$

and

$$\Sigma_{xy} = \rho\sigma w[\tfrac{1}{3}uC + \tfrac{1}{3}vD - (w^2 + \tfrac{1}{9}uv)CD]/(1-w^2CD). \tag{2.11}$$

The matrix, S, of longitudinal propagators is now given by

$$S = (P^{-1} - \Sigma)^{-1} \tag{2.12}$$

where P^{-1} is the matrix of free inverse propagators obtained from equation (2.2),

$$P^{-1}=\begin{pmatrix} p^2+r_1+\frac{1}{2}\rho^2u+\frac{1}{2}\sigma^2w & \rho\sigma w \\ \rho\sigma w & p^2+r_2+\frac{1}{2}\sigma^2v+\frac{1}{2}\rho^2w \end{pmatrix} \tag{2.13}$$

and Σ is the matrix of self-energy parts:

$$\Sigma=\begin{pmatrix} \Sigma_x & \Sigma_{xy} \\ \Sigma_{xy} & \Sigma_y \end{pmatrix}. \tag{2.14}$$

The matrix $P^{-1}-\Sigma$ may be simplied by observing that in zeroth order in $1/n$

$$r_T=r_1+\tfrac{1}{6}\rho^2u+\tfrac{1}{2}\sigma^2w+\tfrac{1}{6}umK(r_T)+\tfrac{1}{2}wnK(\tilde{r}_T) \tag{2.15}$$

and

$$\tilde{r}_T=r_2+\tfrac{1}{6}\sigma^2v+\tfrac{1}{2}\rho^2w+\tfrac{1}{6}vnK(\tilde{r}_T)+\tfrac{1}{2}wmK(r_T). \tag{2.16}$$

Using (2.15) and (2.16) in (2.13), (2.14), (2.9) and (2.10) gives

$$P^{-1}-\Sigma=\begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.17}$$

where

$$a=p^2+r_T+\rho^2(\tfrac{1}{3}uC-1)(w^2D-\tfrac{1}{3}u)/(1-w^2CD) \tag{2.18}$$

$$c=p^2+\tilde{r}_T+\sigma^2(\tfrac{1}{3}vD-1)(w^2C-\tfrac{1}{3}v)/(1-w^2CD) \tag{2.19}$$

and

$$b=\rho\sigma w(\tfrac{1}{3}uC-1)(\tfrac{1}{3}vD-1)/(1-w^2CD). \tag{2.20}$$

Substituting (2.17) in (2.12), gives the longitudinal propagators

$$S_x=a^{-1}/(1-b^2a^{-1}c^{-1}) \tag{2.21}$$

$$S_y=c^{-1}/(1-b^2a^{-1}c^{-1}) \tag{2.22}$$

and

$$S_{xy}=-ba^{-1}c^{-1}/(1-b^2a^{-1}c^{-1}). \tag{2.23}$$

The calculations of § 3 will also require a knowledge of the vertices Γ_x, Γ_y, Γ_{xy} and Γ_{yx} shown in figure 5, where the bubble chains may be mixtures of $\boldsymbol{x}_\perp$ and $\boldsymbol{y}_\perp$ bubbles:

$$\Gamma_x=\rho(\tfrac{1}{9}u^2C+w^2D-\tfrac{2}{3}w^2uCD)/(1-w^2CD)-\tfrac{1}{3}\rho u \tag{2.24}$$

$$\Gamma_y=\sigma(\tfrac{1}{9}v^2D+w^2C-\tfrac{2}{3}w^2vCD)/(1-w^2CD)-\tfrac{1}{3}\sigma v \tag{2.25}$$

$$\Gamma_{xy}=-\sigma w(\tfrac{1}{3}uC-1)(\tfrac{1}{3}vD-1)/(1-w^2CD) \tag{2.26}$$

and

$$\Gamma_{yx}=-\rho w(\tfrac{1}{3}uC-1)(\tfrac{1}{3}vD-1)/(1-w^2CD). \tag{2.27}$$

In addition we shall need the bubble chain contributions to Σ_x, Σ_y, and Σ_{xy} which we shall refer to as $\hat{\Sigma}_x$, $\hat{\Sigma}_y$ and $\hat{\Sigma}_{xy}$:

$$\hat{\Sigma}_x=\rho^2(\tfrac{1}{9}u^2C+w^2D-\tfrac{2}{3}w^2uCD)/(1-w^2CD) \tag{2.28}$$

$$\hat{\Sigma}_y=\sigma^2(\tfrac{1}{9}v^2D+w^2C-\tfrac{2}{3}w^2vCD)/(1-w^2CD) \tag{2.29}$$

and

$$\hat{\Sigma}_{xy}=\Sigma_{xy}. \tag{2.30}$$

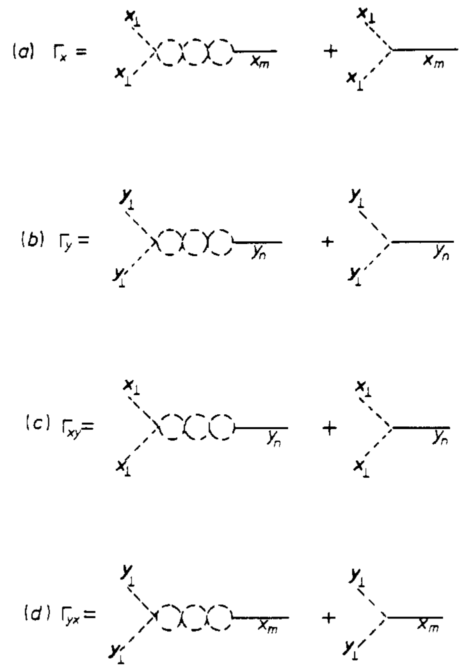

Figure 5. Vertex diagrams connecting a longitudinal propagator to a transverse loop.

3. Equation of state in mixed x–y phase

In this section, the equation of state in the mixed x–y phase is calculated from the free energy functional of equation (2.2) in the limit where the number of components m of $\boldsymbol{x}$ and the number of components n of $\boldsymbol{y}$ become large in such a way that m/n remains finite. For the purposes of the calculation the couplings, u, v and w are taken of order $1/n$ and the order parameters ρ and σ are taken of order $\surd n$. The calculation is performed to next to leading order in $1/n$.

We consider only the case where the gap in temperature between the two phase transitions is large so that ρ may be taken large. The question we aim to answer is whether the transition from the x-ordered to the mixed x–y phase has standard $\mathrm{O}(n)$ symmetric critical indices, or whether the nature of the transition is modified by the long range forces due to the massless Goldstone modes produced by the first transition. The more difficult question of what happens when the second transition occurs before the fluctuations from the first transition have died away, is not discussed.

The loop diagrams contributing to the equation of state to next to leading order in $1/n$ are shown in figure 6. (The equation of state for the Heisenberg model has been calculated at this order by Brézin and Wallace (1973).) The longitudinal propagators shown may be off-diagonal. The equation of state is obtained from the two conditions

$$\langle x_m \rangle = 0 \tag{3.1}$$

and

$$\langle y_n \rangle = 0. \tag{3.2}$$

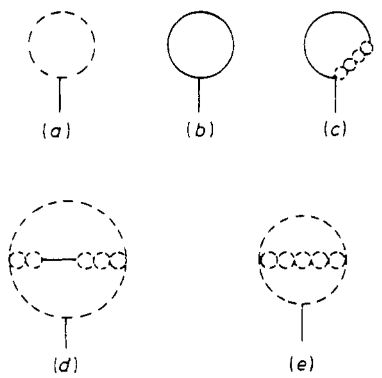

Figure 6. Diagrams for the equation of state. The external full line is either x_m or y_n, and the internal full lines are either x_m propagators, or y_n propagators, or off-diagonal x_m to y_n propagators, in zeroth order in $1/n$. The broken lines are $\boldsymbol{x}_\perp$ or $\boldsymbol{y}_\perp$ propagators, in zeroth order in $1/n$.

Taking the diagrams in order, condition (3.1) gives

$$\begin{aligned} r_1 - r_{1c} &+ \tfrac{1}{6}u(\rho^2-\rho_c^2) + \tfrac{1}{2}w\sigma^2 + \tfrac{1}{6}um\int_p (p^2+r_T)^{-1} + \tfrac{1}{2}wn\int_p (p^2+\tilde{r}_T)^{-1} \\ &+ \tfrac{1}{2}u\int_p S_x(p) + \tfrac{1}{2}w\int_p S_y(p) + \sigma\rho^{-1}w\int_p S_{xy}(p) \\ &-\rho^{-2}\int_p S_x(p)\hat{\Sigma}_x(p) - \rho^{-2}\int_p S_{xy}(p)\hat{\Sigma}_{xy}(p) \\ &+\tfrac{1}{6}um\int_{q,k} (q^2+r_T)^{-2}\{[(k+q)^2+r_T]^{-1}-(k^2+r_T)^{-1}\} \\ &\times(S_x(k)\Gamma_x^2(k)+S_y(k)\Gamma_{xy}^2(k)+S_{xy}(k)\Gamma_x(k)\Gamma_{xy}(k)) \\ &+\tfrac{1}{2}wn\int_{q,k} (q^2+\tilde{r}_T)^{-2}\{[(k+q)^2+\tilde{r}_T]^{-1}-(k^2+\tilde{r}_T)^{-1}\} \\ &\times(S_x(k)\Gamma_{yx}^2(k)+S_y(k)\Gamma_y^2(k)+S_{xy}(k)\Gamma_y(k)\Gamma_{yx}(k)) \\ &+\tfrac{1}{6}um\rho^{-2}\int_{q,k} (q^2+r_T)^{-2}\{[(k+q)^2+r_T]^{-1}-(k^2+r_T)^{-1}\}\hat{\Sigma}_x(k) \\ &+\tfrac{1}{2}wn\sigma^{-2}\int_{q,k} (q^2+\tilde{r}_T)^{-2}\{[(k+q)^2+\tilde{r}_T]^{-1}-(k^2+\tilde{r}_T)^{-1}\}\hat{\Sigma}_y(k)=0. \end{aligned} \tag{3.3}$$

In writing down equation (3.3) the x-ordering field h_ρ has been taken to be zero, and so we must put

$$r_T = 0. \tag{3.4}$$

However, the y-ordering field h_σ is taken to be non-zero, and

$$\tilde{r}_T = h_\sigma/\sigma. \tag{3.5}$$

Equation (3.3) has been subtracted on the critical line for the x-ordered to mixed x–y transition where

$$\tilde{r}_T = \sigma^2 = 0, \qquad \rho^2 = \rho_c^2, \qquad r_1 = r_{1c} \qquad \text{and} \qquad r_2 = r_{2c}. \tag{3.6}$$

This subtraction has been done explicitly for the terms not involving integrals, but, for brevity, it is to be understood for the remaining terms.

Similarly, condition (3.2) gives

$$r_2 - r_{2c} + \tfrac{1}{6}v\sigma^2 + \tfrac{1}{2}w(\rho^2 - \rho_c^2) - h_\sigma/\sigma + \text{(integral terms)} = 0 \tag{3.7}$$

where the integral terms for equation (3.7) may be generated from equation (3.3) by the substitution

$$x \leftrightarrow y, \qquad u \leftrightarrow v, \qquad m \leftrightarrow n, \qquad \rho \leftrightarrow \sigma, \qquad r_T \leftrightarrow \tilde{r}_T. \tag{3.8}$$

Again, these terms are to have a subtraction on the critical line of equation (3.6) understood.

In the critical region for the transition from the x-ordered phase to the mixed x–y phase, $\tilde{r}_T$ and σ^2 are small, whereas ρ^2 is large, provided the two phase transitions are well separated in temperature, as we shall assume. Thus, we shall require equation (2.4) for $p^2 \gg r$, in which case

$$I(p, r) \approx SB(\tfrac{1}{2}\epsilon, 2 - \tfrac{1}{2}\epsilon)B(1 - \tfrac{1}{2}\epsilon, 1 - \tfrac{1}{2}\epsilon)\tfrac{1}{2}p^{-\epsilon} \tag{3.9}$$

where r is either r_T or $\tilde{r}_T$, and

$$S = 2\pi^{2-\frac{1}{2}\epsilon}/(2\pi)^{4-\epsilon}\Gamma(2 - \tfrac{1}{2}\epsilon). \tag{3.10}$$

Apart from the terms of zeroth order in $1/n$, the dominant terms in the critical region are those which are logarithmically more singular than σ^2 or $\tilde{r}_T^{1-\frac{1}{2}\epsilon}$ as σ and $\tilde{r}_T$ tend to zero. To find which terms are in this category, we need the behaviour of $S_y(p)$, $\Gamma_y(p)$, $\hat{\Sigma}_y(p)$ etc in the limit $p \ll 1$, but $p^2 \gg \tilde{r}_T$ and $p^{2-\epsilon} \gg \sigma^2$. In this limit, we see from the equations of § 2 that:

$$S_x(p) \approx (p^2 + r_T + 2\rho^2 I^{-1}(p)/m)^{-1} \tag{3.11}$$

$$S_y(p) \approx (p^2 + \tilde{r}_T + 2\sigma^2 I^{-1}(p)/n)^{-1} \tag{3.12}$$

$$S_{xy}(p) \approx -36\rho\sigma w I^{-2}(p)/mn(uv - qw^2) \times (p^2 + r_T + 2\rho^2 I^{-1}(p)/m)^{-1}(p^2 + \tilde{r}_T + 2\sigma^2 I^{-1}(p)/n)^{-1}. \tag{3.13}$$

Also

$$\Gamma_x(p) \approx -2\rho I^{-1}(p)/m \tag{3.14}$$

$$\Gamma_y(p) \approx -2\sigma I^{-1}(p)/n \tag{3.15}$$

$$\Gamma_{xy}(p) \approx -36\sigma w I^{-2}(p)/mn(uv - 9w^2) \tag{3.16}$$

$$\Gamma_{yx}(p) \approx -36\rho w I^{-2}(p)/mn(uv - 9w^2) \tag{3.17}$$

and

$$\hat{\Sigma}_x(p) \approx \tfrac{1}{3}\rho^2 u - 2\rho^2 I^{-1}(p)/m \tag{3.18}$$

$$\hat{\Sigma}_y(p) \approx \tfrac{1}{3}\sigma^2 v - 2\sigma^2 I^{-1}(p)/n \tag{3.19}$$

$$\hat{\Sigma}_{xy}(p) \approx \rho\sigma w - 36\rho\sigma w I^{-2}(p)/mn(uv - 9w^2). \tag{3.20}$$

Detailed inspection of the (subtracted) integrals of equations (3.3) and (3.7) now shows that none of the integrals involving S_x, S_{xy}, Γ_{xy} or Γ_{yx} is important in the critical region. With the aid of (3.11) to (3.20) equations (3.3) and (3.7) simplify in the critical region to

$$r_1 - r_{1c} + \tfrac{1}{6}u(\rho^2 - \rho_c^2) + \tfrac{1}{2}w\sigma^2 + \tfrac{1}{2}wn \int_p (p^2 + \tilde{r}_T)^{-1} + \tfrac{1}{2}w \int_p S_y(p)$$

$$+ \tfrac{1}{2}wn \int_{q,k} (q^2 + \tilde{r}_T)^{-2}\{[(k+q)^2 + \tilde{r}_T]^{-1} - (k^2 + \tilde{r}_T)^{-1}\}S_y(k)\Gamma_y^2(k)$$

$$- w \int_{q,k} (q^2 + \tilde{r}_T)^{-2}\{[(k+q)^2 + \tilde{r}_T]^{-1} - (k^2 + \tilde{r}_T)^{-1}\}I^{-1}(k) = 0 \tag{3.21}$$

and

$$r_2 - r_{2c} + \tfrac{1}{6}v\sigma^2 + \tfrac{1}{2}w(\rho^2 - \rho_c^2) + \tfrac{1}{6}vn \int_p (p^2 + \tilde{r}_T)^{-1} + \tfrac{1}{6}v \int_p S_y(p)$$

$$+ \tfrac{1}{6}vn \int_{q,k} (q^2 + \tilde{r}_T)^{-2}\{[(k+q)^2 + \tilde{r}_T]^{-1} - (k^2 + \tilde{r}_T)^{-1}\}S_y(k)\Gamma_y^2(k)$$

$$- \tfrac{1}{3}v \int_{q,k} (q^2 + \tilde{r}_T)^{-2}\{[(k+q)^2 + \tilde{r}_T]^{-1} - (k^2 + \tilde{r}_T)^{-1}\}I^{-1}(k) = 0. \tag{3.22}$$

Using (3.21) to eliminate $\rho^2 - \rho_c^2$ from (3.22) gives the equation of state

$$(r_2 - r_{2c}) - 3w(r_1 - r_{1c})/u - h_\sigma/\sigma + (\tfrac{1}{6}v - 3w^2/2u)(\sigma^2 + L) = 0 \tag{3.23}$$

where

$$L = n \int_p (p^2 + \tilde{r}_T)^{-1} + \int_p S_y(p)$$

$$+ n \int_{q,k} (q^2 + \tilde{r}_T)^{-2}\{[(k+q)^2 + \tilde{r}_T]^{-1} - (k^2 + \tilde{r}_T)^{-1}\}S_y(k)\Gamma_y^2(k)$$

$$- 2 \int_{q,k} (q^2 + \tilde{r}_T)^{-2}\{[(k+q)^2 + \tilde{r}_T]^{-1} - (k^2 + \tilde{r}_T)^{-1}\}I^{-1}(k). \tag{3.24}$$

The integrals in equation (3.24) are understood to be subtracted on the critical line of equation (3.6), and $S_y(k)$ and $\Gamma_y(k)$ are now given by equations (3.12) and (3.15).

Neglecting h_σ/σ in the critical region and writing

$$t = [r_2 - r_{2c} - 3w(r_1 - r_{1c})/u]\tfrac{1}{6}v(\tfrac{1}{6}v - 3w^2/2u) \tag{3.25}$$

equation (3.23) may be written as

$$t + \tfrac{1}{6}v\sigma^2 + \tfrac{1}{6}vL = 0 \tag{3.26}$$

which is identical with the equation of state obtained by Brézin and Wallace (1973) for the n-component Heisenberg ferromagnet.

4. Conclusions

It has been shown in the previous section using the $1/n$ expansion correct to order $1/n^2$ that, when the two phase transitions of figure 1 are well separated in temperature, the second phase transition has ordinary $O(n)$ symmetric critical indices despite the presence of Goldstone modes in the x-ordered phase. This happens because all the Feynman diagrams involving Goldstone boson internal lines turn out to be less important in the infrared limit than those involving only soft components of the order parameter as internal lines. It may be conjectured that this is a general result and that the critical exponents for a phase transition between two ordered phases are always determined solely by the components of the order parameter which are going soft at the transition.

An example of such a phase transition is the transition between the A_1 phase and the A phase of superfluid ^3He. In the A_1 phase there are three Goldstone bosons and at the A_1 to A transition two components of the order parameter are going soft. Jones *et al* (1976) assumed that the transition was controlled by all five fields and this resulted in an effective Hamiltonian with no stable renormalization group fixed points and probably a first order phase transition. The conclusions of the present paper suggest that the nature of the transition is determined only by the two components of the order parameter which are going soft, and the transition is then second order with O(2) symmetric critical exponents (the same as for the λ line in liquid ^4He).

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